CONSTRUCTIONS OF TRIANGULAR AND QUADRANGULAR POLYHEDRA OF INSCRIBABLE TYPE

E. Jucovič - S. Ševec - M. Trenkler

Abstract. Procedures of construction of infinite families of triangular and quadrangular polyhedra of inscribable type are presented.

1. A polyhedron (i.e. the convex 3-dimensional hull of a finite number of points in the Euclidean 3-space) $\mathbf{P}$ is said to be *of inscribable type* if there exists a polyhedron $\mathbf{P}^*$ combinatorially isomorphic with $\mathbf{P}$ and a sphere γ that all vertices of $\mathbf{P}^*$ belong to γ. The problem of the existence of polyhedra of noninscribable type goes back at least to Steiner [8]. (For references s. Grünbaum-Shephard [4].) First, sufficient conditions for polyhedra not to be of inscribable type defining large classes of polyhedra have been stated by Steinitz [9] and Grünbaum [3]. In last time new results have been obtained toward characterizations of polyhedra of inscribable type. In Hodgson, Rivin and Smith [6] such a characterisation for general polyhedra is given in terms of whether the polyhedra support certain edge-weightings. Another characterisation for a special class of quadrangular polyhedra is contained in Jucovič-Ševec [7]. Essential sufficient conditions for a polyhedron to be of inscribable type have been presented by Dillencourt and Smith [1], [2].

The aim of this paper is to yield such conditions for triangular and quadrangular polyhedra which are not directly subsumed by the papers of Dillencourt and Smith. The proofs are performed by constructing infinite families of polyhedra having a certain structure.

The basic idea of our constructions is as follows: Let $\mathbf{P}$ be a polyhedron with all vertices belonging to a sphere γ, and S a subgraph of the graph $G(\mathbf{P})$ of the polyhedron $\mathbf{P}$ (i.e. the graph formed by the vertices and edges of $\mathbf{P}$). The vertices, edges and faces incident with S are transformed according to a rule τ so that another polyhedron $\mathbf{P}'$ with graph $G(\mathbf{P}') = \tau G(\mathbf{P})$ with all vertices belonging to γ is created. (The elements of $\mathbf{P}$ not incident with S remain unchanged.) So a new polyhedron of inscribable type is obtained. The polyhedron $\mathbf{P}'$ is constructed so that its graph $G(\mathbf{P}')$ contains a subgraph isomorphic to S which allows to proceed the construction in an analogous way.

2. We start with quadrangular polyhedra. The sufficient condition we intend to prove is

1991 *Mathematics Subject Classification.* 52B10.

Key words and phrases. Polyhedron, Inscribable type.
Mathematica Slovaca **47**(1997), 313-317

Theorem 1. *A quadrangular polyhedron is of inscribable type if its graph arises from the graph of the cube by the successive applying of the transformation τ_1 in Fig. 1.*

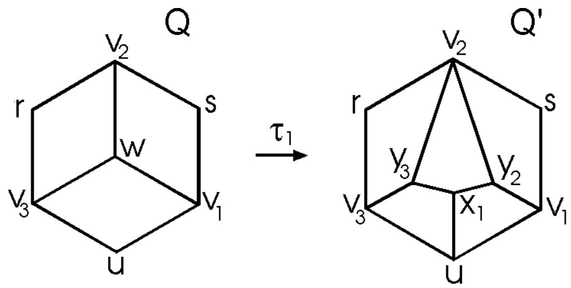

FIGURE 1

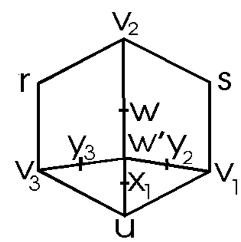

FIGURE 2

Proof. Let us consider a quadrangular polyhedron $\mathbf{P}$ all vertices of which belong to the sphere γ and whose graph $G(\mathbf{P})$ contains a graph isomorphic to Q in Fig.1 as its subgraph. (From now on we will not distinguish the subgraph on the polyhedron and its image on the figures - this should cause no confusions.) Let $\mathbf{w}$ be a trivalent vertex of Q as indicated on Fig.1. On the ray $\mathbf{v}_2\mathbf{w}$ outside of γ near to $\mathbf{w}$ such a point $\mathbf{w}'$ is chosen that the segments $\mathbf{w}'\mathbf{u}, \mathbf{w}'\mathbf{v}_1$ and $\mathbf{w}'\mathbf{v}_3$ intersect γ in three mutually different points $\mathbf{x}_1, \mathbf{y}_2$ and $\mathbf{y}_3$, respectively. Fig.2 depicts the changed subgraph. Take notice of the fact that any choice of the point $\mathbf{w}'$ forces the 4-tuples $(\mathbf{v}_3, \mathbf{y}_3, \mathbf{x}_1, \mathbf{u})$ and $(\mathbf{u}, \mathbf{x}_1, \mathbf{y}_2, \mathbf{v}_1)$ to be coplanar. And moreover, because $\mathbf{w}'$ is on the ray $\mathbf{v}_2\mathbf{w}$, the 4-tuples $(\mathbf{v}_2, \mathbf{y}_3, \mathbf{v}_3, \mathbf{r})$ and $(\mathbf{v}_2, \mathbf{s}, \mathbf{v}_1, \mathbf{y}_2)$ are coplanar, too. It holds: The convex hull of

$$V(\mathbf{P}) - \{\mathbf{w}\} \cup \{\mathbf{x}_1, \mathbf{y}_2, \mathbf{y}_3\} \qquad (V(\mathbf{P}) \text{ is the vertex set of } \mathbf{P})$$

is a polyhedron $\mathbf{N}$ with all vertices belonging to γ. The edges and faces of the polyhedron $\mathbf{P}$ not incident to the vertex $\mathbf{w}$ are not changed, they are edges and faces of $\mathbf{N}$. We must still show that the graph of $\mathbf{N}$ is $\tau_1 G(\mathbf{P})$ i.e. that it contains Q' as subgraph. To do this it clearly suffices to show that the vertices $\mathbf{v}_2, \mathbf{y}_3, \mathbf{x}_1, \mathbf{y}_2$ belong to the same face of $\mathbf{N}$, i.e. that vertices $\mathbf{y}_3, \mathbf{y}_2$ as well as $\mathbf{v}_2, \mathbf{x}_1$ are not joined by an edge.

This can be done so that Steinitz's [9] condition for a polyhedron $\mathbf{X}$ not to be of inscribable type is used: *If the vertex-set $V(\mathbf{X})$ of the polyhedron $\mathbf{X}$ has a subset T with $|T| \geq \frac{|V(\mathbf{X})|}{2}$ such that no two of its vertices are joined by an edge, and in case $|T| = \frac{|V(\mathbf{X})|}{2}$ in the set $V(\mathbf{X}) - T$ there exist two vertices joined by an edge - then $\mathbf{X}$ is not of inscribable type.*

Let us return to our polyhedron $\mathbf{N}$. As the polyhedron $\mathbf{P}$ is quadrangular, all circuits of $G(\mathbf{P})$ have even lengths and the graph $G(\mathbf{P})$ is bichromatic (cf. Harary [5]). As $\mathbf{P}$ is inscribed in γ, by the above theorem of Steinitz, both colour-classes of $G(\mathbf{P})$ have equal numbers of vertices. On a quadrangular face adjacent vertices do not belong to the same colour-class. Let all vertices of $\mathbf{N}$, - with the exception of $\mathbf{x}_1, \mathbf{y}_2$ and $\mathbf{y}_3$ retain the colours they have in $\mathbf{P}$. It is possible to colour the vertices $\mathbf{x}_1, \mathbf{y}_2, \mathbf{y}_3$ so that a (possibly non-regular) 2-colouring of $G(\mathbf{N})$ with equal numbers of vertices in both colour-classes appears. However, the existence of an edge $\mathbf{y}_3\mathbf{y}_2$ or $\mathbf{v}_2\mathbf{x}_1$ in the graph of the polyhedron $\mathbf{N}$ would by the mentioned theorem of Steinitz, mean the non-inscribability of $\mathbf{N}$ - a contradiction.

So in fact the graph of **N** has been obtained by performing operation τ_1 on the graph of the polyhedron **P**, $\mathbf{N} = \mathbf{P}'$. In the graph of the polyhedron $\mathbf{P}'$ we have got again Q as its subgraph, therefore the procedure τ_1 can be performed once more, and so on. The proof of Theorem 1 is concluded by recalling that the cube is of inscribable type.

3. An analogous reasoning is fruitfull if we intend to construct triangular polyhedra of inscribable type. Fig.3 presents transformations of graphs of triangular polyhedra of inscribable type into more complicated ones retaining inscribability.

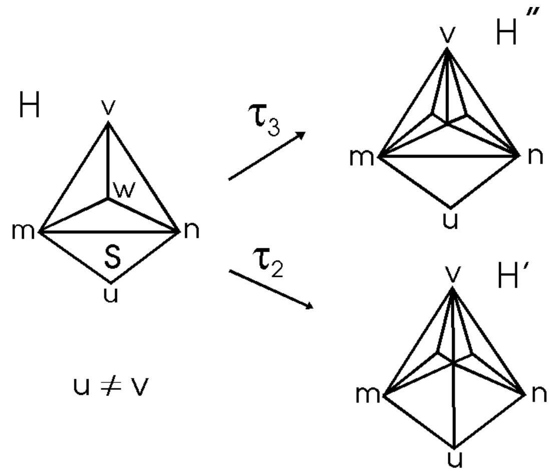

FIGURE 3

Theorem 2. *A triangular polyhedron is of inscribable type if its graph arises from the graph of the bipyramid with 6 faces (Fig.4) by successive applying the transformations* τ_2 *or* τ_3 *depicted in Fig.3.*

Proof. Let the graph of the triangular polyhedron **P** contain as proper subgraph the graph H in Fig.3, and let all vertices of **P** belong to the sphere γ. The plane containing the face s intersects γ in a circuit c decomposed by the vertices **m**, **n** into two arcs. Choose inside of that arc that does not contain the vertex **u** three poins $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ and join them with the vertex **v** (Fig.5.) The vertex **w** and the edges incident with it are removed. A polyhedron **Q** with all vertices belonging to γ appears with vertex-set $V(\mathbf{Q}) = \{V(\mathbf{P}) - \{\mathbf{w}\}\} \cup \{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ and a hexagonal face $f = \mathbf{uny}_1\mathbf{y}_2\mathbf{y}_3\mathbf{m}$. Crucial in next steps of the construction is the following

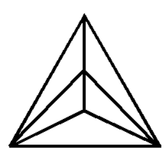

FIGURE 4

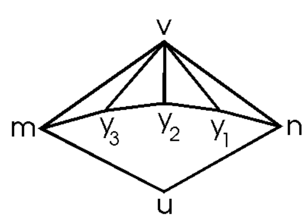

FIGURE 5

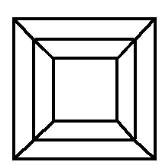

FIGURE 6

Lemma. *(Dillencourt-Smith [2]) Let* **M** *be a polyhedron of inscribable type with a non-bipartite graph. Add to this graph as new edge the diagonal of some face of* **M**. *The resulting graph is isomorphic to the graph of a polyhedron of inscribable type.*

Let us return to polyhedron **Q** which clearly does not admit a bipartite graph. Add to the graph $G(\mathbf{Q})$ sucessively the diagonals $\mathbf{y}_2\mathbf{m}, \mathbf{y}_2\mathbf{n}$ and $\mathbf{y}_2\mathbf{u}$ of the hexagonal face f. At the end we get a graph containing H' as subgraph. Proceeding analogously as just before but inserting the diagonal **mn** instead of $\mathbf{u}\mathbf{y}_2$ we get a graph containing H'' as subgraph. Both these graphs, by Lemma are realisable as graphs of polyhedra of inscribable type. The polyhedra are triangular and contain the graph H as subgraph. Thus the construction can proceed.

4. Remarks

1. We can show that neither Theorem 1 defines all quadrangular polyhedra of inscribable type nor does Theorem 2 so with the triangular ones. As examples for quadrangular polyhedra can serve the polyhedra constructed as follows: On the surface of the sphere consider two mutually perpendicular great circles and divide every of the four half-circles obtained into $m \geq 3$ congruent arcs. The convex hull of the dividing points is a polyhedron of inscribable type. To made it quadrangular replace the two quadruples of triangles by two quadrangles. For $m = 4$, Fig.6 presents the polyhedron. All polyhedra obtained in this way can be employed for performing transformation τ_1 at the beginning of the construction. The transformation τ_1 can be employed for constructing other polyhedra of inscribable type having even-gonal faces only.

2. Further triangular polyhedra of inscribable type are constructed if the hexagonal face f of **Q** in the proof of Theorem 2 is otherwise decomposed by diagonals into triangles or if on the arc of c with end-points **m**, **n** any number $\neq 3$ of new vertices is set. Of cours also non-triangular polyhedra of inscribable type can be constructed in this way.

3. It is desirable to find analogous constructions of pentagonal polyhedra of inscribable type or to prove that the dodecahedron is the only such polyhedron.

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